Introduction to Probability Theory
Problem set #3

1. Imagine you wish to determine a length. Which do you prefer, a set of ten measurements with a resolution of 1 mm or just one measurement with a resolution of 0.2 mm? *Hint: You don't need Estimation Theory to solve this problem.*

2. Let $P(x)$ ($x \in \mathbb{R}$) denote a probability distribution with mean μ. Let $\{x_i\}_{i=1...N}$ denote a data sample drawn according to P. Show that

$$\frac{N}{(N-1)(N-2)} \sum_{i=1}^{N} (x_i - \bar{x})^3$$

is an unbiased estimator for $\langle (x-\mu)^3 \rangle$, where $\bar{x} = \frac{1}{N}(x_1 + \ldots + x_N)$ denotes the sample mean.

3. Let $\{x_i\}_{i=1...N}$ denote a data sample, where each x_i is drawn according to a probability distribution density

$$f_\theta(x) = \frac{1}{\theta} e^{-x/\theta}, x \in \mathbb{R}^+,$$

where θ is an *unknown* parameter. For $r > 0$, find a maximum likelihood estimator for Proba$[X \leq r]$, where X is distributed according to f_θ.

4. We have seen that quantum states allow to violate Bell inequalities. For instance, if we consider the EPR state

$$\frac{1}{\sqrt{2}} \left(|\mathbf{1/2}, +1/2\rangle_A |\mathbf{1/2}, -1/2\rangle_B - |\mathbf{1/2}, -1/2\rangle_A |\mathbf{1/2}, +1/2\rangle_B \right), \tag{1}$$

then it is possible to violate the inequality

$$|\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle| \leq 2. \tag{2}$$

for the choice of observables

$$A_1 = \sigma_A^z, \qquad A_2 = \sigma_A^x,$$

$$B_1 = -\frac{\sigma_B^z + \sigma_B^x}{2}, \qquad B_2 = \frac{\sigma_B^z - \sigma_B^x}{2}.$$

Assume that each correlator is estimated with just *two* EPR pairs. That is, we measure $A_i \otimes B_j$ on a first pair and obtain some outcome w_{ij}^1, measure the same observable on some other pair, get a result w_{ij}^2 and decide on $\frac{1}{2}(w_{ij}^1 + w_{ij}^2) = C_{ij}$ as an estimate for the correlator $\langle A_i B_j \rangle$.

Consider the assertion "Nature is consistent with the above Bell inequality" as a null hypothesis, H_0. If we choose not to reject H_0 if

$$|C_{11} + C_{12} + C_{21} - C_{22}| \leq 2, \tag{3}$$

what is the probability to make a type II error? *Hint: First solve a slightly different problem. Consider the case where each correlator is estimated with just **one** EPR pair.*